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LETTER TO THE EDITOR

A super extension of WKI integrable systems

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Abstract. A super extension of WKI integrable systems proposed using a super-sl(2, R) valued soliton connection 1-form. Classes of super-integrable nonlinear evolution equations are found.

The super integrable systems in general and the super extension of the standard integrable systems in two-dimensional spacetime have been recently investigated including the following cases: supersymmetric sine-Gordon system (Girardello and Sciuto 1978), supersymmetric Liouville model (Chaichian and Kulish 1978), supersymmetric σ models (D'Auria and Sciuto 1980), supersymmetric Toda lattices (Ol'shanetsky 1983), a super extension of KdV system (Kupershmidt 1984a, b), super integrable systems of Lax type (Kupershmidt 1984c) and supersymmetric KP hierarchy (Manin and Radul 1985).

Very recently, a super extension of the AKNS scheme has been proposed which exploits a super-sl(2, R) algebra valued soliton connection 1-form and a class of super integrable nonlinear evolution equations is obtained containing the super extension of nonlinear Schrödinger equation (Gürses and Oğuz 1985a). Furthermore, by introducing an internal symmetry, an O(N) extended super AKNS scheme is constructed (Gürses and Oğuz 1985b).

In this work we study the super extension of the integrable systems given by Wadati, Konno and Ichikawa (WKI) (1979a, 1979b). With this aim we first introduce a super soliton connection 1-form

$$\Omega = e_i\theta_i + q_a\pi_a = \begin{pmatrix} \theta_0 & \theta_1 & \pi_1 \\ \theta_2 & -\theta_0 & \pi_2 \\ \pi_2 & -\pi_1 & 0 \end{pmatrix} \tag{1}$$

where e_i ($i = 0, 1, 2$) are bosonic and q_a ($a = 1, 2$) are fermionic generators of super-sl(2, R) algebra (Gürses and Oğuz 1985a). Following the method of WKI (1979b) we parametrise 1-forms, θ_i and π_a, as follows

$$\theta_0 = A(t, x, \lambda)\, dt - i\lambda\, dx \qquad \pi_1 = \alpha(t, x, \lambda)\, dt + \lambda\beta(t, x)\, dx$$

$$\theta_1 = C(t, x, \lambda)\, dt + \lambda r(t, x)\, dx \qquad \pi_2 = \rho(t, x, \lambda)\, dt + \lambda\varepsilon(t, x)\, dx$$

$$\theta_2 = B(t, x, \lambda)\, dt + \lambda q(t, x)\, dx$$

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where λ is a constant spectral parameter, A, B, C (α, ρ) are bosonic (fermionic) super functions and q, r (β, ε) are bosonic (fermionic) super potential as in the case of the super AKNS scheme.

Then the linear eigenvalue equations and the time evolution equations can be written in a unified form

$$\mathrm{d}\Psi+\Omega\Psi=0 \tag{2}$$

where d is an exterior derivative and the Jost function Ψ is $\Psi^{\mathrm{T}}=(\psi_1, \psi_2, \phi)$ with the commuting functions ψ_1, ψ_2 and the anticommuting function ϕ. The integrability of (2) leads to the vanishing curvature 2-form

$$\mathrm{d}\Omega+\Omega\wedge\Omega=0 \tag{3}$$

where $\wedge$ is an exterior product. Namely, Ω is a flat connection. The representation of (3) in terms of superfunctions and potentials is given by

$$A_x+\lambda(rB-qC+\beta\rho-\alpha\varepsilon)=0 \tag{4a}$$

$$B_x+\lambda(-q_t+2\mathrm{i}B+2Aq+2\varepsilon\rho)=0 \tag{4b}$$

$$C_x+\lambda(-r_t-2\mathrm{i}C-2Ar-2\beta\alpha)=0 \tag{4c}$$

$$\alpha_x+\lambda(-\beta_t-\mathrm{i}\alpha-A\beta-C\varepsilon+r\rho)=0 \tag{4d}$$

$$\rho_x+\lambda(-\varepsilon_t+\mathrm{i}\rho+A\varepsilon-B\beta+q\alpha)=0. \tag{4e}$$

These equations yield classes of super integrable nonlinear partial differential equations (NLPDE) by substituting the positive power series of the spectral parameter λ,

$$A=\sum_{n=0}^{N} a_n\lambda^n, \qquad B=\sum_{n=0}^{N} b_n\lambda^n, \qquad C=\sum_{n=0}^{N} c_n\lambda^n,$$

$$\alpha=\sum_{n=0}^{N} \alpha_n\lambda^n, \qquad \rho=\sum_{n=0}^{N} \rho_n\lambda^n,$$

into (4) and by comparing the coefficients of λ^n.

As a first example, let us consider $N=2$. Hence the following nonlinear evolution equations are obtained:

$$q_t=-\frac{H_0}{2}\left[\left(1-\frac{\beta\varepsilon}{1-rq}\right)\left(\frac{q}{(1-rq)^{1/2}}\right)_x+\left(\frac{q\beta\varepsilon}{(1-rq)^{3/2}}\right)_x + \frac{2q\varepsilon}{1-rq}\left(\frac{\beta}{(1-rq)^{1/2}}\right)_x\right]_x-\mathrm{i}H_0\left(\frac{\varepsilon\varepsilon_x}{(1-rq)^{3/2}}\right)_x, \tag{5a}$$

$$r_t=\frac{H_0}{2}\left[\left(1-\frac{\beta\varepsilon}{1-rq}\right)\left(\frac{r}{(1-rq)^{1/2}}\right)_x+\left(\frac{r\beta\varepsilon}{(1-rq)^{3/2}}\right)_x - \frac{2r\beta}{1-rq}\left(\frac{\beta}{(1-rq)^{1/2}}\right)_x\right]_x+\mathrm{i}H_0\left(\frac{\beta\beta_x}{(1-rq)^{3/2}}\right)_x, \tag{5b}$$

$$\varepsilon_t=-H_0\left\{\frac{1}{1-rq}\left[\left(\frac{\varepsilon}{(1-rq)^{1/2}}\right)_x+\frac{(2-qr)\beta\varepsilon\varepsilon_x}{2(1-rq)^{3/2}}+\frac{q\varepsilon}{2}\left(\frac{r}{(1-rq)^{1/2}}\right)_x\right]\right\}_x + \mathrm{i}H_0\left\{\frac{1}{1-rq}\left[\frac{\beta}{2}\left(\frac{q}{(1-rq)^{1/2}}\right)_x+q\left(\frac{\beta}{(1-rq)^{1/2}}\right)_x-\frac{3}{2}\frac{q\beta\beta_x\varepsilon}{(1-rq)^{3/2}}\right]\right\}_x, \tag{5c}$$

$$\beta_t = H_0\left\{\frac{1}{1-rq}\left[\left(\frac{\beta}{(1-rq)^{1/2}}\right)_x + \frac{(2-qr)\beta\varepsilon\beta_x}{2(1-rq)^{3/2}} + \frac{r\beta}{2}\left(\frac{q}{(1-rq)^{1/2}}\right)_x\right]\right\}_x$$
$$+\mathrm{i}H_0\left\{\frac{1}{1-rq}\left[\frac{\varepsilon}{2}\left(\frac{r}{(1-rq)^{1/2}}\right)_x + r\left(\frac{\varepsilon}{(1-rq)^{1/2}}\right)_x - \frac{3}{2}\frac{r\beta\varepsilon_x\varepsilon}{(1-rq)^{3/2}}\right]\right\}_x, \qquad (5d)$$

where H_0 is a commuting constant. Equations ($5a$)-($5d$) are the superextension of the first type integrable NLPDE of WKI (1979b). The special choices are as follows.

(i) The first type WKI coupled NLPDE: $\varepsilon = \beta = 0$, $H_0 = 2\mathrm{i}$.

(ii) Taking $r = k_1\bar{q}$, $\beta = k_2\bar{\varepsilon}$, $H_0 = 2\mathrm{i}$, k_i = real constants ($\bar{k}_i = k_i$) where a bar sign denotes Berezin adjoint operation (Berezin 1966) we obtain

$$q_t = -\mathrm{i}\left[\left(1 - \frac{k_2\bar{\varepsilon}\varepsilon}{1-k_1|q|^2}\right)\left(\frac{q}{(1-k_1|q|^2)^{1/2}}\right)_x + \left(\frac{k_2 q\bar{\varepsilon}\varepsilon}{(1-k_1|q|^2)^{3/2}}\right)_x\right.$$
$$\left. + \frac{2k_2 q\varepsilon}{1-k_1|q|^2}\left(\frac{\bar{\varepsilon}}{(1-k_1|q|^2)^{1/2}}\right)_x\right]_x + \left(\frac{2\varepsilon\varepsilon_x}{(1-k_1|q|^2)^{3/2}}\right)_x \qquad (6a)$$

$$\varepsilon_t = -\mathrm{i}\left\{\frac{1}{1-k_1|q|^2}\left[\left(\frac{2\varepsilon}{(1-k_1|q|^2)^{1/2}}\right)_x + \frac{k_2(2-k_1|q|^2)\bar{\varepsilon}\varepsilon\varepsilon_x}{(1-k_1|q|^2)^{3/2}} + k_1 q\varepsilon\left(\frac{\bar{q}}{(1-k_1|q|^2)^{1/2}}\right)_x\right]\right\}_x$$
$$-\left\{\frac{1}{1-k_1|q|^2}\left[k_2\bar{\varepsilon}\left(\frac{q}{(1-k_1|q|^2)^{1/2}}\right)_x + k_2 q\left(\frac{2\bar{\varepsilon}}{(1-k_1|q|^2)^{1/2}}\right)_x\right.\right.$$
$$\left.\left. - \frac{3k_2^2 q\bar{\varepsilon}\bar{\varepsilon}_x\varepsilon}{(1-k_1|q|^2)^{3/2}}\right]\right\}_x \qquad (6b)$$

with $k_1 = k_2^2$.

(iii) Taking $r = k_1 q^*$, $\beta = k_2\varepsilon^*$, $H_0 = 2\mathrm{i}$, k_1 = real constant ($k_1^* = k_1$) and k_2 = complex constant ($k_2^* = -k_2$) we obtain the above equations (6), where Berezin adjoint operation is replaced by * complex conjugation operation and consequently with the constraint $k_1 = |k_2|^2$.

In the case of $N = 3$ the following super integrable NLPDES arise from (4)

$$q_t = \frac{\mathrm{i}K_0}{4}\left[-Z_x^1 + \frac{\beta\varepsilon}{1-rq}\left(\frac{q_x}{(1-rq)^{3/2}}\right)_x - \frac{2\varepsilon}{1-rq}Y_x^2 + \frac{2q\varepsilon}{1-rq}Y_x^1\right]_x$$
$$+\frac{K_0}{4}\left[\left(\frac{(2+rq)\varepsilon\varepsilon_x - 3q^2\beta\beta_x}{(1-rq)^{5/2}}\right)_x + \frac{2\varepsilon}{1-rq}Q_x^1 - \frac{2q\varepsilon}{1-rq}Q_x^2\right]_x, \qquad (7a)$$

$$r_t = \frac{\mathrm{i}K_0}{4}\left[-Z_x^2 + \frac{\beta\varepsilon}{1-rq}\left(\frac{r_x}{(1-rq)^{3/2}}\right)_x + \frac{2\beta}{1-rq}Q_x^2 - \frac{2r\beta}{1-rq}Q_x^1\right]_x$$
$$+\frac{K_0}{4}\left[\left(\frac{(2+rq)\beta_x - 3r^2\varepsilon\varepsilon_x}{(1-rq)^{5/2}}\right)_x + \frac{2\beta}{1-rq}Y_x^1 - \frac{2r\beta}{1-rq}Y_x^2\right]_x, \qquad (7b)$$

$$\varepsilon_t = \frac{\mathrm{i}K_0}{2}\left[-\frac{1}{1-rq}\left(1 + \frac{(1+rq)\beta\varepsilon}{1-rq}\right)Q_x^1 + \frac{q}{1-rq}\left(1 + \frac{2\beta\varepsilon}{1-rq}\right)Q_x^2\right.$$
$$\left. - \frac{\beta}{2(1-rq)}\left(\frac{(2+qr)\varepsilon\varepsilon_x}{(1-rq)^{5/2}}\right)_x + \frac{q\varepsilon}{2(1-rq)}Z_x^2\right]_x$$

$$+\frac{K_0}{2}\left[-\frac{1}{1-rq}\left(1+\frac{(1+rq)\beta\varepsilon}{1-rq}\right)Y_x^2+\frac{q}{1-rq}\left(1+\frac{2\beta\varepsilon}{1-rq}\right)Y_x^2\right.$$

$$\left.-\frac{q\varepsilon}{2(1-rq)}\left(\frac{(2+qr)\beta\beta_x}{(1-rq)^{5/2}}\right)_x-\frac{\beta}{2(1-rq)}Z_x^1\right]_x, \tag{7c}$$

$$\beta_t=\frac{\mathrm{i}K_0}{2}\left[-\frac{1}{1-rq}\left(1+\frac{(1+rq)\beta\varepsilon}{1-rq}\right)Y_x^1+\frac{r}{1-rq}\left(1+\frac{2\beta\varepsilon}{1-rq}\right)Y_x^2\right.$$

$$\left.+\frac{\varepsilon}{2(1-rq)}\left(\frac{(2+qr)\beta\beta_x}{(1-rq)^{5/2}}\right)_x+\frac{r\beta}{2(1-rq)}Z_x^1\right]_x$$

$$+\frac{K_0}{2}\left[\frac{1}{1-rq}\left(1+\frac{(1+rq)\beta\varepsilon}{1-rq}\right)Q_x^2-\frac{r}{1-rq}\left(1+\frac{2\beta\varepsilon}{1-rq}\right)Q_x^1\right.$$

$$\left.-\frac{r\beta}{2(1-rq)}\left(\frac{(2+qr)\varepsilon\varepsilon_x}{(1-rq)^{5/2}}\right)_x+\frac{\varepsilon}{2(1-rq)}Z_x^2\right]_x \tag{7d}$$

where K_0 is a commuting constant and

$$Q^1=\left(\frac{\varepsilon}{(1-rq)^{3/2}}\right)_x+\frac{\varepsilon_x}{(1-rq)^{3/2}},\qquad Q^2=\left(\frac{r\varepsilon}{(1-rq)^{3/2}}\right)_x+\frac{r\varepsilon_x}{(1-rq)^{3/2}},$$

$$Y^1=\left(\frac{\beta}{(1-rq)^{3/2}}\right)_x+\frac{\beta_x}{(1-rq)^{3/2}},\qquad Y^2=\left(\frac{q\beta}{(1-rq)^{3/2}}\right)_x+\frac{q\beta_x}{(1-rq)^{3/2}},$$

$$Z^1=\frac{q_x}{(1-rq)^{3/2}}+\frac{3q(\beta\varepsilon_x-\beta_x\varepsilon)}{(1-rq)^{5/2}},$$

$$Z^2=\frac{r_x}{(1-rq)^{3/2}}-\frac{3r(\beta\varepsilon_x-\beta_x\varepsilon)}{(1-rq)^{5/2}}.$$

Equations (7*a*)-(7*d*) are the super extension of the second type integrable equations of WKI (1979b). As in the previous case we can list the following choices.

(i) The second type WKI coupled NLPDE: $\varepsilon=\beta=0$, $\mathrm{i}K_0=4$.

(ii) Taking $r=k_1\bar{q}$, $\beta=k_2\bar{\varepsilon}$, $\mathrm{i}K_0=4$, k_i = real constants ($\bar{k}_i=k_i$) (7) reduce to the following two nonlinear equations

$$q_t=\left[-Z_x^1+\frac{k_2\bar{\varepsilon}\varepsilon}{1-k_1|q|^2}\left(\frac{q_x}{(1-k_1|q|^2)^{3/2}}\right)_x-\frac{2\varepsilon}{1-k_1|q|^2}Y_x^2\right.$$

$$\left.+\frac{2q\varepsilon}{1-k_1|q|^2}Y_x^1\right]_x-\mathrm{i}\left[\left(\frac{(2+k_1|q|^2)\varepsilon\varepsilon_x-3k_2^2q^2\bar{\varepsilon}\bar{\varepsilon}_x}{(1-k_1|q|^2)^{5/2}}\right)_x\right.$$

$$\left.+\frac{2\varepsilon}{1-k_1|q|^2}Q_x^1-\frac{2q\varepsilon}{1-k_1|q|^2}Q_x^2\right]_x, \tag{8a}$$

$$\varepsilon_t=\left[-\frac{2}{1-k_1|q|^2}\left(1+\frac{(1+k_1|q|^2)k_2\bar{\varepsilon}\varepsilon}{1-k_1|q|^2}\right)Q_x^1+\frac{2q}{1-k_1|q|^2}\left(1+\frac{2k_2\bar{\varepsilon}\varepsilon}{1-k_1|q|^2}\right)Q_x^2\right.$$

$$\left.-\frac{k_2\bar{\varepsilon}}{1-k_1|q|^2}\left(\frac{(2+k_1|q|^2)\varepsilon\varepsilon_x}{(1-k_1|q|^2)^{5/2}}\right)_x+\frac{q\varepsilon}{1-k_1|q|^2}Z_x^2\right]_x$$

$$-\mathrm{i}\left[-\frac{2}{1-k_1|q|^2}\left(1+\frac{(1+k_1|q|^2)k_2\bar{\varepsilon}\varepsilon}{1-k_1|q|^2}\right)Y_x^2+\frac{2q}{1-k_1|q|^2}\left(1+\frac{2k_2\bar{\varepsilon}\varepsilon}{1-k_1|q|^2}\right)Y_x^1\right.$$

$$-\frac{q\varepsilon}{1-k_1|q|^2}\left(\frac{(2+k_1|q|^2)k_2^2\bar{\varepsilon}\bar{\varepsilon}_x}{(1-k_1|q|^2)^{5/2}}\right)_x-\frac{k_2\bar{\varepsilon}}{1-k_1|q|^2}Z_x^1\Bigg]_x, \qquad (8b)$$

where $k_1=k_2^2$ and

$$Q^1=\frac{1}{k_2}\bar{Y}^1=\left(\frac{\varepsilon}{(1-k_1|q|^2)^{3/2}}\right)_x+\frac{\varepsilon_x}{(1-k_1|q|^2)^{3/2}}$$

$$Q_2^2=k_2\bar{Y}^2=\left(\frac{k_1\bar{q}\varepsilon}{(1-k_1|q|^2)^{3/2}}\right)_x+\frac{k_1\bar{q}\varepsilon_x}{(1-k_1|q|^2)^{3/2}},$$

$$Z^1=\frac{1}{k_1}\bar{Z}^2=\frac{q_x}{(1-k_1|q|^2)^{3/2}}+\frac{3k_2q(\bar{\varepsilon}\varepsilon_x-\bar{\varepsilon}_x\varepsilon)}{(1-k_1|q|^2)^{5/2}}.$$

(iii) Another choice with $r=k_1q^*$, $\beta=k_2\varepsilon^*$, $\mathrm{i}K_0=4$, $k_1=$ real constant $(k_1^*=k_1)$ and $k_2^*=$ complex constant $(k_2^*=-k_2)$ leads to the equations which have the same form as (8) with $k_1=|k_2|^2$ and the Berezin operation is replaced by * complex conjugation.

One may also obtain a super extension of the Harry Dym equation from (7) exploiting a suitable choice of the fields. It is also possible to extend these super integrable systems introducing an internal index to the fermionic generators, q_a^i, $(i=1,2,\ldots,N)$. Hence, in this way, one can increase the number of fermionic fields and obtain $\mathrm{O}(N)$ extended super integrable systems as in the $\mathrm{O}(N)$ extended super AKNS scheme (Gürses and Oğuz 1985b). In this formulation the gauge transformation of the Jost function,

$$\Psi'=S\Psi$$

and of the connection

$$\Omega'=S\Omega S-(\mathrm{d}S)S^{-1}$$

where 3×3 super matrix, S, depends on the spectra parameter λ, corresponds to a Bäcklund transformation for the super NLPDE under consideration (Gürses 1984) when supplemented by the Zakharov-Shabat (1979) reduction procedure.

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